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Investigating Algebraic Procedures using Discussion and Writing

Abstract

This study reports on the implementation of an intermediate algebra curriculum centered on a framework of student-centered questions designed to investigate algebraic procedures. Instructional activities were designed to build discourse in the small-group discussion meetings of the course. Students were assigned writing prompts to emphasize the importance of understanding different aspects of a procedure beyond simple execution of the procedure. Skills-based pre- and post-tests were administered as well as a researcher developed procedural understanding instrument. No significant differences were observed between the treatment and control groups on posttest performance; however the treatment group showed a significant difference in gains from the pretest to the posttest. The treatment group did score significantly higher on the procedural understanding exam suggesting that the treatment was effective without hindering basic skill development.

This research study evaluates the effectiveness of using classroom discussion and student journaling to focus lessons on a series of investigative questions to help students gain a deep, well-connected understanding of algebraic procedures.

Research has been conducted over the last several years that has focused on students' algebraic deficiencies and the development of strategies to combat the problems of skill development and retention. Implementing a practical, instructional framework has yielded some promising results. Hasenbank (2006) developed and implemented a Framework for Procedural Understanding (hereafter referred to as the "Framework") in a college algebra course. This Framework was developed through observations, student interviews, and reflection on available learning research. The goal of this Framework is to enhance student performance and retention by deepening their understanding of algebraic procedures.

This article reports on results of a curriculum and instructional treatment that incorporated the Framework in a developmental intermediate algebra course. This Framework is based upon guidelines that were proposed by National Council of Teachers of Mathematics (NCTM). The teaching and learning of mathematics should strive for the following goals:

1. The student understands the overall goal of the algebraic process and knows how to predict or estimate the outcome.
2. The student understands how to carry out an algebraic process and knows alternative methods and *representations* of the process.
3. The student understands and can *communicate* to others why the process is effective and leads to valid results.
4. The student understands how to evaluate the results of an algebraic process by invoking *connections* with a context or with other mathematics the student knows.
5. The student understands and uses mathematical *reasoning* to assess the relative efficiency and accuracy of an algebraic process compared with alternative methods that might have been used.
6. The student understands why an algebraic process empowers her or him as a mathematical *problem solver* (NCTM, 2001, p. 31, emphasis in original).

Hasenbank re-expressed these guidelines as eight student-centered questions for application in the classroom setting:
1. (a) What is the goal of the procedure, and (b) what sort of answer should I expect?
2. (a) How do I execute the procedure, and (b) what are some other procedures I could use instead?
3. Why is the procedure effective and valid?
4. What connections or contextual features could I use to verify my answer?
5. When is this the "best" procedure to use?
6. What can I use this procedure to do? (Hasenbank, 2006, p. 7-8)

The focus of many developmental algebra courses and students is on 2a of the Framework, "How do I execute the procedure?" A brief review of mathematics education suggests that algebraic expertise does not result from extensive practice alone. For instance, a sole focus on performing procedures results in fragile student knowledge, procedures that are not executed "intelligently," and the rise of systematic errors (Star, 2000). Contemporary literature contends that expertise is a blend of conceptual and procedural, with increases in one leading to advances in the other (Rittle-Johnson, Siegler, & Alibali, 2001). Mathematical procedures that are rooted in conceptual understanding are more easily remembered and applied to new situations (Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992; Kieran, 1992; Van Hiele, 1986). Connections between mathematical concepts and procedures facilitate recall, ease future learning, and allow users to recreate steps that may have been forgotten (Hiebert & Carpenter, 1992).

The NCTM (1989, 1991, 2000) has repeatedly emphasized practices to establish mathematical discourse communities and outlined the roles of students and teachers in these communities. Discourse, most broadly defined, is any specific act of communication, including verbal and nonverbal, synchronic or asynchronous, and with others or with oneself (Ben-Yehuda et al., 2006; Sfard, 2001). Goos (2004) found that students believed that explaining their thinking was a means to strengthen and evaluate their understanding. These explanations from students came as responses to questions similar to Framework questions such as "How could we verify this?" and "What was the reason for completing this procedure?"

Developmental students need the benefit of learning in the manner described above and the Framework provides a method for doing so. Through several iterations and evaluations of Framework-based curricula, the researcher realized that engaging students and holding them accountable for this learning was critical (Harper, 2007). Adding a written journaling component to the curriculum was made to aid in this student accountability. The NCTM, American Mathematical Association of Two-Year Colleges (AMATYC), and Mathematical Association of America (MAA) all advocate that writing should be incorporated into the mathematics classroom yet many teachers fail to even consider implementation due to fears of feasibility (Seto & Meel, 2006).

The "Framework curriculum" for the research reported here is an attempt to account for the findings from the literature reviewed above. Specifically, the Framework curriculum used the student-centered question to build conceptual connections while learning to execute algebraic procedures. Class discussion and student writing assignments were used to both build a community of discourse and more fully engage the students to look beyond just the execution of algebraic procedures.

Research Questions

After a series of pilot projects, the Framework curriculum was implemented in an intermediate algebra course at a public, 4-year university during the spring semester of 2010. We seek here to answer the following questions:

1. Do students learning through the Framework curriculum have the same level of Framework-based procedural understanding as students in the traditional courses?
2. Do students learning through the Framework curriculum have the same level of *procedural skill* as students in the traditional courses?

Methodology

This study utilized a quasi-experimental design with the assignment of each section of a developmental algebra course into either the control or treatment group. The students registered for the course without being aware of the experiment. 14 sections of approximately 24 students each were taught in the spring of 2010 of which 3 sections were selected as the treatment group. The treatment sections were selected based on the researcher's teaching availability.

Course Structure

All sections attended 50-minute lectures with the professor two days per week. These lectures were essentially identical, differing only by infrequent questions asked by individual students during the lectures. Each section also attended discussion sections with a graduate student teaching assistant two days a week between lecture days. At these discussion sections, students sat at tables in groups of four and did activities, detailed below, based on the previous lecture given by the professor.

The control group discussion sections were taught using examples, worksheets, and short participation assignments, which primarily involved calculations or execution of the current procedure being taught. The treatment group discussion sections were taught using the Framework curriculum, with an emphasis on the six student-centered questions for each algebraic procedure of interest. Worksheets were seldom used, and the students were assigned writing prompts instead. Figures 1 and 2 below shows he differences between the two groups for the same algebra topic.

Figure 1. Differences between Control and Treatment Sections in Activities for Students Learning about Expressions with Exponents

Procedure of Interest: Simplifying Expressions with Exponents	
Control Section	Treatment Section
• Review of exponent rules • Worksheet on negative exponents • Worksheet on mathematical heteronyms • Items from worksheet graded for participation points	• Discussion of the goal of the procedure • Students expanded expressions to show why the exponent rules are valid • Proof of quotient rule for exponents graded for participation points

Figure 2. Differences between Control and Treatment Sections in Activities for Students Learning about Finding the Slope of a Line

Procedure of Interest: Finding the Slope of a Line	
Control Section	Treatment Section
• Students calculated slope given multiple points on a graph • Students looked at linear models and found information based on the slope of the models • Questions about the data being modeled graded for participation points	• Discussion of the slope of a line as a rate of change • Used a computer program to explain the validity of the "rise over run" formula • Questions about linear models graded for participation points

The Framework Curriculum

The instructional activities in the Framework curriculum specified beginning each of the discussion sessions by introducing the students to the 6 questions. Each lesson would begin with the announcement of the algebraic procedure of interest for the day. The instructor would lead the class in a discussion of the student-centered Framework questions as they related to that procedure of interest. Usually during one of the two meetings each week, the class would close with a writing prompt with the students turning in their response on a 3x5 card to be graded for participation points (see figure 3).

Figure 3. Examples of writing prompt questions for topics above

Procedure of Interest: Simplifying Expressions with Exponents
Writing Prompt: A classmate simplified the expression $(4x^3)^2$ and thinks that the answer is $16x^6$. Show your classmate how to verify that $(4x^3)^2 = 16x^6$ without using the formulas (because that would just repeat your classmate's work.)
Procedure of Interest: Finding the Slope of a Line
Writing Prompt: Explain why we graph functions, and what the slope of the graph tells us about the function.

From the perspective of the instructor, implementation of the Framework curriculum was not difficult. The framework questions were posted daily on a Smart Board, and the students were engaged in a discussion of each question. This discussion was sometimes centered around activities using graphing utilities and computer algebra software on the Smart Board. Once all six questions were answered, the examples of execution of the procedures were performed. These were alternated between examples done on the board by the instructor and examples done by the students in groups.

Research Design

The independent variable was exposure to the student-centered Framework questions and to the writing prompts that were integrated into the Framework curriculum. The dependent variables of interest were the students' levels of procedural understanding and procedural skill. The null hypothesis was that there was no difference in student understanding or skills between the two groups.

The student's procedural understanding was measured using a 16-question instrument developed by the researchers. This Framework test was given to all treatment and control sections. The instrument underwent an initial validity review by Hasenbank, the original author of the student-centered Framework questions. The instrument was also piloted during the curriculum pilot during the prior spring semester of 2009. Each question was designed to demonstrate knowledge of a specific algebraic procedure, with regards to one of the Framework questions.

All questions were free-response, and were evaluated on a 5-point scale by the researchers. The scale ranged from 4, indicating complete understanding, to 1, indicating no understanding, based on a rubric developed by the researchers. A score of 0 was given for no response. An inter-rater reliability test was conducted and the results are presented below.

In addition to the procedural understanding instrument, all students were given a 17 item pretest at the beginning of the semester. The pretest questions were skills-based questions, requiring execution of a typical intermediate algebra procedure. These same 17 questions were embedded in the final exam, and the difference in pre/post scores was used to measure a student's change in procedural skill.

Results

In addition to validity checks of the Framework-based instrument described above, the researchers conducted a reliability study on the evaluation of student responses on the instrument. Then independent samples tests were conducted to compare the treatment and control groups on procedural understanding and procedural skill.

Instrument Reliability

The researchers considered percent agreement, Pearson correlations and Cohen's kappa to determine inter-rate reliability of evaluating students on the Framework instrument. The results are presented in Table 1 below. Two items (questions 3 and 8) with a kappa less than .4 were not used in the final analysis of the research question. The other questions were included in the data analysis to compare the treatment and control groups.

Table 1.

Measure of Inter-rater Reliability on the Framework Instrument

Instrument Item	Percent Agreement	Pearson Correlation	Cohen's Kappa
1	70.1%	.758	.553
2	69.1%	.706	.407
3	52.9%	.702	.357
4	77.9%	.886	.691
5	64.7%	.818	.494
6	63.2%	.897	.542

7	97.1%	.943	.958
8	52.9%	.770	.366
9	75.0%	.777	.653
10	72.1%	.887	.675
11	86.8%	.831	.778
12	66.2%	.921	.564
13	67.6%	.868	.576
14	72.1%	.862	.610
15	72.1%	.929	.639
16	75.0%	.894	.614

Procedural Understanding

There were 14 items that were reliably scored on the Framework instrument. The researchers examined the differences in means between the treatment and control group (see Table 2). We see a significant difference in 5 of the 14 items evaluated, as well as in the total scores.

Table 2.

Comparing Groups on Framework Instrument to Measure Procedural Understanding

Instrument Item	Control Mean N = 214	Treatment Mean N = 59	t df = 271	p
1	2.0140	2.4068	3.810	.000
2	1.8832	2.3220	3.802	.000
4	2.8131	2.6207	-1.224	.222
5	2.9299	3.0508	-.654	.514
6	1.5280	1.7288	1.355	.176
7	2.6542	2.8983	1.417	.158
9	2.5701	2.7627	1.366	.173
10	1.6916	2.2712	3.386	.001
11	3.2249	3.2034	-.155	.877
12	2.5654	2.8475	1.551	.122
13	2.0000	2.2034	1.413	.159
14	1.3146	1.6610	3.606	.000

15	2.5841	2.8305	1.581	.115
16	1.3271	1.5763	2.223	.027
Total	**30.2290**	**33.6271**	**3.370**	**.001**

Procedural Skill

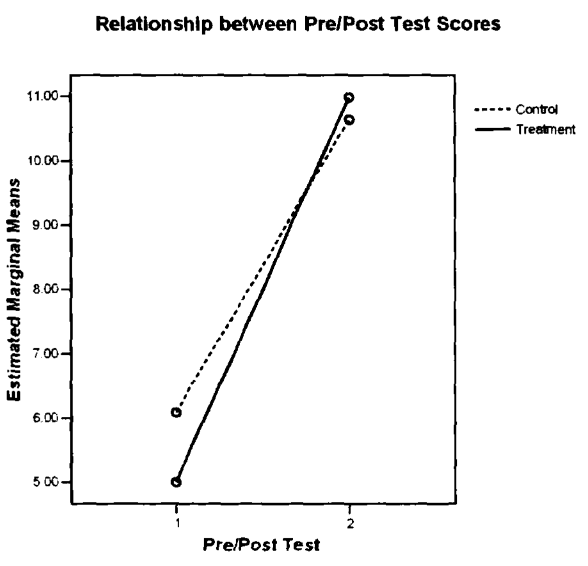
Independent samples t-tests were conducted to compare the treatment and control groups with scores on the pre and post test as well as the gain made from pre to post test (see Table 3). There was a significant difference observed between the pre-test scores in favor of the treatment group. A significant different was not observed in the post-test scores, or the final exam scores, but a significant difference was measured in the pre/post-gain, in favor of the treatment group. This is shown in the graph in Figure 4 below.

Table 3.

Comparing Groups on Pre and Post Tests to Measure Procedural Skill

Measure	Treatment Mean N = 59	Control Mean N = 214	t	p	Correlation with Framework Test
Pre-Test	5.0951	6.0498	2.514	.013	.263
Post-Test	10.7895	10.5942	-.406	.685	.555
Pre/Post Difference	5.9819	4.5510	-2.669	.008	.282
Final Exam	64.0351	63.0865	-.408	.684	.646

Figure 4.



Limitations

There were several limitations in this research, revolving around instructor and treatment group selection and the research instrument. Random assignment to treatment and control groups was not possible. One instructor was chosen to teach the sections comprising the treatment group. The sections were then determined by the schedule of this instructor. The instructor was also purposively chosen and was the only instructor to teach the Framework curriculum. It would be preferable to use multiple instructors, each teaching both treatment and control section, to control for teacher effect.

The researchers took care to develop a valid and reliable instrument and scoring procedure but certain questions on the research instrument proved difficult to evaluate consistently, and therefore were removed from the data. Refinement of the instrument would be crucial before the research was repeated.

Conclusions

One possible conclusion of these results is that students exposed to the Framework curriculum had a deeper understanding of algebraic procedures than students using the traditional curriculum. The significantly higher achievement of the treatment group on the Framework assessment may also indicate that the treatment students improved in their capacity to communicate and write about mathematics. While it is to be expected that the students who were exposed to the writing prompts would outperform the control group on a free response instrument, a more interesting result was observed on the pre/post-testing. The students were given a pre-test during the first week of class, and the same questions were embedded in the final exam. The treatment sections scored significantly lower on the pre-test, but by the end of the semester, scored insignificantly different from the control group. The pre/post gain was significantly greater for the treatment group than the control group. This evidence suggests that both the treatment and control groups demonstrated a comparable level of procedural skill at the conclusion of the course.

We conclude that, while the treatment did not result in significantly higher performance on the final exam, the improved understanding and communication skills shown by the treatment group on the research instrument appears to make the treatment worthwhile. In addition, the treatment students were able to perform at a comparable level of skill compared to the control group, showing that the treatment was not detrimental to these students. This is an important finding because there is a clear shift of focus in the Framework curriculum from practicing execution of algebraic procedures to investigating, discussing, and writing about different aspects of these same procedures.

National organizations such as NCTM, AMATYC, and MAA have recommended incorporating writing into the mathematics classroom; however Seto & Meel (2006) indicated that many were hesitant to do so. The above findings suggest that a community of discourse can be created in the classroom and that time writing about mathematics can replace time spent on traditional, skill-based activities without hindering attainment of the required skills to successfully pass the course.

The Framework curriculum instructor in this study recorded his efforts to implement the alternate curriculum and reported that teaching in this different style during the spring semester was not more difficult than teaching the course in the traditional style during the previous fall semester. In fact, the treatment instructor chose to utilize the framework questions and writing prompts in subsequent introductory level mathematics classes in his future teaching assignments with some success. Students in this study and in future courses taught by the treatment instructor were hesitant to write about mathematics. Our study suggests that, despite this hesitation, it may improve their understanding and it does not appear to harm their performance on traditional testing measures.

Implications

Further Research

The results of this study indicate that further curriculum development should occur to more effectively utilize writing in developmental mathematics. The course structure of intermediate algebra at this institution was such that students met in large lecture sessions twice a week with a professor and in small discussion sections of 24 students with a graduate teaching assistant. The Framework curriculum in this study was implemented in just the small group sections. One reason for this was that treatment and control sections were both attending the same large lecture session. With a larger scale implementation, the Framework questions would also permeate the lectures even though there is little student interaction in this setting. The design would embed the Framework questions as a natural part of the curriculum rather than highlighting them as a separate part of each lesson.

In the future, it would be important repeat this experiment, using multiple instructors, to address one of the limitations described above. Further refinement of the instrument to measure procedural understanding would also be important to better characterize the benefits of using this Framework and writing exercises in teaching mathematics.

This study adds to the body of research supporting the use of writing in mathematics courses. Developmental mathematics should be taught to emphasize both efficiency in executing procedures as well as understanding the concepts and mathematical connections behind them. This study contributes to the feasibility and attainability of teaching in this way. More investigations of this type would be beneficial to the developmental mathematics community.

Teaching and Developmental Education

The utilization of investigative questions (such as the Framework questions) and writing assignments is feasible and effective in enhancing learning in developmental mathematics. In this study, the existing course format was large lecture with small group meetings. This type of curriculum could be applied to developmental mathematics courses of any style, be it lecture, small group discussion, online, computer-assisted, or a hybrid of these. Technology is greatly impacting the delivery of mathematics courses with an increase in online offerings as well as self-paced, computer-assisted courses. In both face-to-face and online offerings, these discussion questions and writing prompts can be used to deepen students' understanding of algebraic procedures.

The Framework curriculum used in this study exposes developmental students to the idea that mathematics is more than just algorithmic procedures, a set of rules and skills to remember and follow. The role of developmental mathematics is to prepare students for college-level mathematics. This preparation will naturally focus on how to correctly and efficiently execute algebraic procedures, but is should also encourage students to look for connections among and a deeper understanding of procedures. This understanding can also be helpful to students in seeing how mathematics can relate to other disciplines they are studying. This approach could be used in conjunction with other developmental teaching techniques to bring mathematics to all levels of students.

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